Elliptic r-matrix systems, affine Lie algebras, and projective representations of the braid group of the torus(GEOMETRIC ASPECTS OF INFINITE ANALYSIS)

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Elliptic r-matrix systems, affine Lie algebras, and projective representations of the braid group of the torus

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1. Classical r-matrices

<u>Def</u> A classical r-matrix is a meromorphic function $r(z)$ in a neighborhood of the origin in $\mathbb{C}$ with values in $\mathfrak{g} \otimes \mathfrak{g}$ (where $\mathfrak{g}$ is a simple Lie algebra) which satisfies the classical Yang-Baxter equation:

$$[r_{12}(z_1-z_2), r_{13}(z_1-z_3)] + [r_{12}(z_1-z_2), r_{23}(z_2-z_3)] + [r_{13}(z_1-z_3), r_{23}(z_2-z_3)] = 0$$

in $U(\mathfrak{g})^{\otimes 3}$. Here $r_{12} = r \otimes 1$, $r_{23} = 1 \otimes r$, $r_{13} = S_{12}(1 \otimes r) S_{12}$, S_{12} - the permutation of 1-st and 2-nd factor.

2. Why look at classical r-matrices?

(A) $R_h(z)$ - quantum R-matrix (solution to the quantum Yang-Baxter equation) $\Rightarrow$ if $R_h(z) = 1 + h\, r(z) + O(h^2)$, then $r(z)$ is a classical r-matrix (Sklyanin).

(B) Let $V_1, \cdots, V_n$ be $\mathfrak{g}$-modules.
$\pi_i : \mathfrak{g} \longrightarrow End(V_i)$ - the corresponding Lie algebra homomorphisms, $r_{ij}(z) = \pi_i \otimes \pi_j (r(z))$
$r_{ij} \hookrightarrow End(V_1 \otimes \cdots \otimes V_n)$.
Consider the 1-form

$$\omega = \frac{1}{\kappa} \sum_{i<j} r_{ij}(z_i - z_j)\, d(z_i - z_j), \quad \kappa \in \mathbb{C}$$

Observe $d\omega = 0$.
<u>claim</u> $r(z)$ satisfies the classical YB equation $\Longleftrightarrow$ $[\omega, \omega] = 0$ (Cherednik)
This condition is equivalent to the zero curvature condition $d\omega + [\omega, \omega] = 0$ for the connection $\nabla = d + \omega$.

3. Classical r-matrix systems

We see that $r(z)$ satisfies the classical YB equation iff the system of differential equations $df + \omega f = 0$ with respect to a $V_1 \otimes \cdots \otimes V_n$ - valued function $f(z_1, \dots, z_n)$ has $\dim(V_1 \otimes \cdots \otimes V_n)$ linearly independent solutions. This system is called the r-matrix system corresponding to r.

4. Classification of classical r-matrices

In 1982 A. Belavin and V. Drinfeld showed that if $r(z)$ is a classical r-matrix which is invertible as a map $\mathfrak{g}^* \to \mathfrak{g}$ for a generic $z \in \mathbb{C}$ then $r(z)$, up to an equivalence relation is of one of the three types: rational, trigonometric, and elliptic. Elliptic non-degenerate r-matrices exist only for $\mathfrak{g} = \mathfrak{sl}_N$, and they were found by Belavin and Sklyanin. Trigonometric r-matrices exist for any $\mathfrak{g}$ and were classified by Belavin and Drinfeld in 1982. There is no classification of rational r-matrices.

5. Examples of r-matrices

A An example of a rational r-matrix is the Yang's r-matrix

$$r_{rat}(z) = \frac{\Omega}{z},$$

$\Omega = \sum_i X_i \otimes X_i$, $\{X_i\}$ is an orthonormal base of $\mathfrak{g}$ with respect to the Killing form $\langle\, ,\, \rangle$. The corresponding r-matrix system is the system of Knizhnik-Zamolodchikov equation,

$$\varkappa \frac{\partial f}{\partial z_i} = \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j} f, \quad \Omega_{ij} = \pi_i \otimes \pi_j(\Omega)$$

B Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the Cartan decomposition. Let e_α be a $\mathfrak{h}$-eigenbasis of $\mathfrak{n}^+$, f_α be the dual basis of $\mathfrak{n}^-$ with respect to $\langle\, ,\, \rangle$, $\alpha \in \Delta^+$. Let $\{H_i\}$ be an orthonormal base of $\mathfrak{h}$ with respect to $\langle\, ,\, \rangle$.
Set

$$\Omega^+ = \sum_\alpha e_\alpha \otimes f_\alpha + \frac{1}{2} \sum_i H_i \otimes H_i$$

$$\Omega^- = \sum_\alpha f_\alpha \otimes e_\alpha + \frac{1}{2} \sum_i H_i \otimes H_i$$

Then the simplest trigonometric r-matrix is

$$r_{trig}(z) = \frac{\Omega^+ e^z + \Omega^-}{e^z - 1}$$

The corresponding r-matrix system is the system of trigonometric KZ equations.

<u>Fact</u> : Trigonometric KZ can be reduced to rational (usual) KZ by a simple transformation.

<u>C</u> Let $\mathfrak{g} = sl_N(\mathbb{C})$, and let $\beta, \gamma \in \mathrm{Aut}(\mathfrak{g})$ be two commuting elements with no common invariant nonzero elements in $\mathfrak{g}$. Any such pair can be reduced by a conjugation to

$$\beta = \mathrm{Ad}\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & \\ 0 & 0 & \ddots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \qquad \gamma = \mathrm{Ad}\begin{pmatrix} 1 & & & 0 \\ & \varepsilon^{-1} & & \\ & & \varepsilon^{-2} & \\ 0 & & & \ddots \ \varepsilon^{1-N} \end{pmatrix}$$

where ε is a primitive N-th root of unity. Let $\zeta(z|\tau)$ be the Weierstrass elliptic ζ-function:

$$\zeta(z|\tau) = \frac{1}{z} + \lim_{M\to\infty} \sum_{\substack{-M \le m, p \le M \\ m^2+p^2>0}} \left[\frac{1}{z-m-p\tau} + \frac{z}{(m+p\tau)^2} \right]$$

$$(\tau \in \mathbb{C}^\times)$$

Then

$$r_{ell}(z) = \Omega\, \zeta(z|\tau) +$$

$$\sum_{\substack{0 \leq m,\, p \leq N-1 \\ m^2+p^2>0}} (1 \otimes \beta^p \gamma^{-m})(\Omega) \times$$

$$\left[\zeta\left(z - \frac{m+p\tau}{N} \Big| \tau\right) - \zeta\left(-\frac{m+p\tau}{N} \Big| \tau\right) \right]$$

is an elliptic r-matrix. Belavin and Drinfeld showed that any elliptic r-matrix is (up to a constant factor) equivalent to this one. It is the quasiclassical limit of the Baxter-Belavin quantum R-matrix of the (generalised) 8-vertex model. $r_{ell}(z)$ is uniquely defined by the following conditions:

1) $r_{ell}(z)$ is meromorphic in $\mathbb{C}$ and has no other singularities than simple poles at $(m+p\tau)/N$, $m, p \in \mathbb{Z}$ with residues $(1 \otimes \beta^p \gamma^{-m})(\Omega)$;

2) $r_{ell}(z)_{12} = -r_{ell}(-z)_{21}$
(unitarity condition)

The lattice of periods of $r_{ell}(z)$ is spanned by 1 and τ, and its lattice of poles is spanned by $1/N$ and τ/N.

6. Statement of the problem

Rational and trigonometric KZ equations come from representation theory. They are satisfied by the correlation functions of the Wess-Zumino-Witten conformal field theory, which are matrix elements of intertwining operators between modules over affine Lie algebras. A natural question is:

Does there exist such an interpretation for the elliptic r-matrix system?

7. Twisted realization of the affine Lie algebra $sl_N(\mathbb{C})$

$$\mathfrak{g} = sl_N(\mathbb{C}), \qquad \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} \cdot c$$

$$[a(t) + \lambda c,\ b(t) + \mu c] = [a(t), b(t)] + \frac{1}{2\pi i} \oint_{|t|=1} \langle a'(t), b(t) \rangle\, t^{-1} dt \cdot c$$

(normalization of $\langle\ ,\ \rangle$: $\langle \tilde{\alpha}, \tilde{\alpha} \rangle = 2$, where $\tilde{\alpha}$ is the highest root).

Let $\varepsilon = e^{\frac{2\pi i l}{N}}$ be a primitive N-th root of unity. Set

$$C = \begin{pmatrix} 1 & & & 0 \\ & \varepsilon^{-1} & & \\ & & \ddots & \\ 0 & & & \varepsilon^{1-N} \end{pmatrix} \qquad \begin{array}{l} \gamma = \operatorname{Ad} C \\ \gamma(a) = C a C^{-1} \end{array}$$

$\hat{\mathfrak{g}}_\gamma \subset \hat{\mathfrak{g}}$: $\hat{\mathfrak{g}}_\gamma = \{ a(t) + \lambda c \in \hat{\mathfrak{g}} \mid a(\varepsilon t) = \gamma(a(t)) \}$

<u>Basis of $\mathfrak{g}$</u> : E_{ij}, $i \neq j$, $1 \leq i, j \leq N$,
$H_i = E_{ii} - E_{i+1,\, i+1}$,
$1 \leq i \leq N-1$.

<u>Basis of $\hat{\mathfrak{g}}$</u> : $E_{ij} \otimes t^m$, $H_i \otimes t^m$, c

<u>Basis of $\hat{\mathfrak{g}}_\gamma$</u> : $E_{ij} \otimes t^{j-i+mN}$,
$H_i \otimes t^{mN}$, c

<u>Lemma</u> $\hat{\mathfrak{g}}_\gamma \simeq \hat{\mathfrak{g}}$

<u>Pf</u>. $\phi(E_{ij} \otimes t^{j-i+mN}) = E_{ij} \otimes t^m$
$\phi(H_i \otimes t^{mN}) = H_i \otimes t^m$, $m \neq 0$
$\phi(H_i) = H_i - \frac{c}{N}$, $\phi(c) = \frac{c}{N}$

gives an isomorphism.

8. Modules over $\hat{\mathfrak{g}}_\gamma$

Polarization : $\hat{\mathfrak{g}}_\gamma = \hat{\mathfrak{g}}_\gamma^+ \oplus (\mathfrak{h} \oplus \mathbb{C}\cdot c) \oplus \hat{\mathfrak{g}}_\gamma^-$

$\hat{\mathfrak{g}}_\gamma^+$ is spanned by $a(t) \in \hat{\mathfrak{g}}_\gamma$ with $a(0)=0$.

$\hat{\mathfrak{g}}_\gamma^-$ is spanned by $a(t) \in \hat{\mathfrak{g}}_\gamma$ with $a(\infty)=0$.

Verma modules over $\hat{\mathfrak{g}}_\gamma$:

$$M_{\lambda,k} = \mathrm{Ind}_{\hat{\mathfrak{g}}_\gamma^+ \oplus \mathfrak{h} \oplus \mathbb{C}\cdot c}^{\hat{\mathfrak{g}}_\gamma} X_{\lambda,k}$$

$X_{\lambda,k}$ - 1-dim $\hat{\mathfrak{g}}_\gamma^+ \oplus \mathfrak{h} \oplus \mathbb{C}\cdot c$-module generated by $v_{\lambda,k} \in X_{\lambda,k}$

$$\hat{\mathfrak{g}}_\gamma^+ v_{\lambda,k} = 0, \qquad h\, v_{\lambda,k} = \langle \lambda, h\rangle v_{\lambda,k}$$

$$c\, v_{\lambda,k} = k\, v_{\lambda,k}.$$

(Under the isomorphism ϕ, $M_{\lambda,k}$ goes to $M_{\Lambda,K}$, standard Verma module over $\hat{\mathfrak{g}}$ with $\Lambda = \lambda + k\rho$, $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$, $K = Nk$.

Evaluation representations of $\hat{\mathfrak{g}}_\gamma$:

V - finite dimensional $\mathfrak{gl}_N$ - module

$V(z) = V \otimes \mathbb{C}[z, z^{-1}]$

$\hat{\mathfrak{g}}$ acts in $V(z)$:

$$[(a(t) + \lambda c)\cdot v](z) = a(z)\, v(z), \quad z \in \mathbb{C}^\times.$$

$C: V \to V$ is defined since

$C \in GL_N$ $\left(C = \begin{pmatrix} 1 & & 0 \\ & \varepsilon^{-1} \ddots & \\ 0 & & \varepsilon^{1-N} \end{pmatrix} \right)$.

$$V_C(z) \overset{\text{def}}{=} \{ v(z) \in V(z) \mid v(\varepsilon z) = C\, v(z) \}$$

<u>Observe</u>: $V_C(z)$ is stable under the action of $\hat{\mathfrak{g}}_\gamma \subset \hat{\mathfrak{g}}$.

<u>Fact</u>: Under the isomorphism ϕ, $V_C(z)$ over $\hat{\mathfrak{g}}_\gamma$ goes to $V(z)$ over $\hat{\mathfrak{g}}$.

<u>Grading</u>: Let ∂ be the operator of principal gradation:

$$[\partial, a(t) + \lambda c] = t\, a'(t), \quad a(t) + \lambda c \in \hat{\mathfrak{g}}_\gamma$$

Set $\tilde{\mathfrak{g}}_\gamma = \hat{\mathfrak{g}}_\gamma \oplus \mathbb{C}c$. Operator ∂ naturally acts on $\hat{\mathfrak{g}}_\gamma$-modules:

On $V_C(z)$: $\quad \partial v(z) = z \dfrac{d\, v(z)}{dz}$

On $M_{\lambda, k}$: $\quad \partial X = -\dfrac{\langle \lambda, \lambda \rangle}{2(k+1)} + \deg(X)$

where X is homogeneous.

$z^{\Delta} V_C(z)$ - the same module as $V_C(z)$, but ∂ acts as

$$\partial v = z\frac{dv}{dz} + \Delta v \qquad (\Delta \in \mathbb{C})$$

9. Intertwiners

We want to study $\tilde{\mathfrak{g}}_\gamma$-intertwining operators

$$\Phi(z) : M_{\lambda,k} \longrightarrow M_{\nu,k} \hat{\otimes} z^{\Delta} V_C(z)$$

where $\Delta = \dfrac{\langle \nu, \nu \rangle - \langle \lambda, \lambda \rangle}{2(k+1)}$

and $M \hat{\otimes} V$ denotes the completed tensor product.

Theorem $\Phi(z)$ (for generic parameters) are in 1-1 correspondence with vectors in V of weight $\lambda - \nu$, and $\Phi(z)$ is uniquely determined by the highest matrix coefficient $\langle v^*_{\nu,k}, \Phi(z) v_{\lambda,k} \rangle \in V^{\lambda-\nu}$

10. Products of intertwiners

Suppose we have

$$\Phi_1(z_1) : M_{\lambda_1, k} \longrightarrow M_{\lambda_0, k} \hat{\otimes} z_1^{\Delta_1} V_1(z_1)$$

$$\Phi_2(z_2) : M_{\lambda_2, k} \longrightarrow M_{\lambda_1, k} \hat{\otimes} z_2^{\Delta_2} V_2(z_2)$$

$$\cdots$$

$$\Phi_n(z_n) : M_{\lambda_n, k} \longrightarrow M_{\lambda_{n-1}, k} \hat{\otimes} z_n^{\Delta_n} V_n(z_n)$$

<u>Question</u> : Can we make sense of the product $\Phi_1(z_1) \cdots \Phi_n(z_n)$?
(assuming now $z_1, \dots, z_n \in \mathbb{C}^*$)

<u>Answer</u> : Yes, we can if

$$|z_1| > |z_2| > \cdots > |z_n|$$

Then every matrix element of this product is a convergent power series.

Thus, we have an operator

$$\phi = \phi_1 \phi_2 \cdots \phi_n : M_{\lambda_n, k} \to \hat{M}_{\lambda_0 k} \otimes V_1 \otimes \cdots \otimes V_n$$

($\hat{M}$ is the completion of M with respect to the principal gradation) which depends on $z_1, \dots, z_n$ and is defined

in the region $|z_1| > |z_2| > \cdots > |z_n|$.
Note that it will, in general, be a multivalued function since it will have a factor $z_1^{\Delta_1} \cdots z_n^{\Delta_n}$

$$\left(\Delta_i = \frac{\langle \lambda_{i-1}, \lambda_{i-1} \rangle - \langle \lambda_i, \lambda_i \rangle}{2(k+1)} \right).$$

In KZ theory we study the highest matrix element $\langle v^*_{\lambda_0,k}, \phi v_{\lambda_n k} \rangle$ of ϕ and prove that it satisfies the trigonometric KZ equation.

11. The B-operator

In the case $\mathfrak{g} = sl_N$, the Dynkin diagram of $\hat{\mathfrak{g}}$ admits an outer automorphism of order N. It corresponds to an outer automorphism $\beta : \hat{\mathfrak{g}}_\gamma \longrightarrow \hat{\mathfrak{g}}_\gamma$. It restricts to the previously introduced β on $\mathfrak{h} \subset \hat{\mathfrak{g}}_\gamma$.
Define the map

$$B : M_{\lambda,k} \longrightarrow M_{\beta(\lambda),k}$$

$$(\beta(\lambda)(h) = \lambda(\beta^{-1}(h)))$$

by

$$B v_{\lambda,k} = v_{\beta(\lambda),k}$$

$$\beta a v = \beta(a) B v, \quad a \in \hat{\mathfrak{g}}_\gamma.$$

Then $B^N = 1$.

12. Traces

Assume that $\lambda_n = \beta(\lambda_0)$. Fix q, $|q|<1$

Consider the trace

$$F(z_1, \cdots, z_n | q)$$
$$= \mathrm{Tr}|_{M_{\lambda_0,k}} \left(\phi_1(z_1) \cdots \phi_n(z_n) B q^{-\partial} \right)$$

<u>claim</u> This is convergent when

$$|z_1| > |z_2| > \cdots > |z_n| > |q z_1|$$

So, it defines a holomorphic function in this region. (Notation: $q = e^{2\pi i \tau}$).

13. Main theorem

$$(k+1)\, z_j \frac{\partial F}{\partial z_j} = \sum_{j \neq l} \frac{1}{2\pi i N} r_{ell} \left(\frac{\log z_j - \log z_l}{2\pi i} \Big| N\tau \right)_{jl} F$$

This is equivalent to the elliptic r-matrix system – we just have to make a change of variable $y_j = c \log z_j$.

<u>claim</u> For generic parameters, if we take all possible $\lambda_0, \lambda_1, \ldots, \lambda_n$ and all possible intertwiners with these λ_i, we get very nice basis of the space of solutions of the elliptic r-matrix system.

14. The moduli equation.

The defined traces satisfy one more differential equation – of the form

$$q \frac{\partial F}{\partial q} = (\text{something known}) \cdot F$$

Before we write it down, introduce some notations.

New variables: $\quad y_j = \dfrac{N \log z_j}{2\pi i}$

$\tau = \dfrac{\log q}{2\pi i}$, $\quad \varkappa = k+1$. $\quad \rho(y|\tau) = \frac{1}{N^2} r_{ell}\left(\frac{y}{N} \middle| N\tau\right)$

Then, we have the elliptic r-matrix system

$$\varkappa \frac{\partial F}{\partial y_\ell} = \sum_{j \neq \ell} \rho_{\ell j}(y_\ell - y_j \,|\, \tau) F$$

Let $\quad s(y|\tau) = \displaystyle\int_0^y \frac{\partial \rho(x|\tau)}{\partial \tau}\, dx$

(the integral is well-defined since all singularities of $\frac{\partial \rho}{\partial \tau}$ are second order poles with residue 0). Also, $L(\tau)$ is a $\mathfrak{g} \otimes \mathfrak{g}$ valued "modular" function which for $\mathfrak{g} = sl_2$ equals

$$2\pi i\, L(\tau) = -\sum_{m\geq 0} \frac{q^{2m+1}(1+q^{4m+2})}{(1-q^{4m+2})^2}(e\otimes e + f\otimes f)$$
$$-\sum_{m\geq 0}\frac{2q^{4m+2}}{(1-q^{4m+2})^2}(e\otimes f + f\otimes e)$$
$$+\ \frac{1}{2}\left(\frac{1}{8} + \sum_{m>0}\frac{q^{2m}}{(1+q^{2m})^2}\right) h\otimes h$$

Then,

Theorem The function F satisfies the differential equation

$$\varkappa \frac{\partial F}{\partial \tau} = \sum_{i,j=1}^{n} L_{ij}(\tau) F + \sum_{j<i} s_{ij}(y_i - y_j|\tau) F$$

(here $L_{ii}(\tau)$ is defined according to the rule $(a\otimes b)_{ii} = \pi_i(a)\,\pi_i(b)$, $a, b \in \mathfrak{g}$, $\pi_i : \mathfrak{g} \longrightarrow \mathrm{End}(V_i)$)

So, we get an extended system of $n+1$ consistent differential equation. Let us call it the elliptic KZ equations.

Corollary $\mathrm{Tr}|_{M_{0,k}}(Bq^{-\partial}) = 1$

15. Examples

$n = 1$ — then we only have the moduli equation, but it is already interesting.
$V = V_1 = $ 2-d representation of sl_2.
Then we have two possible operators:

$$\phi^+(z) : M_{-1/2,k} \longrightarrow M_{1/2,k} \otimes V$$

$$\phi^-(z) : M_{-1/2,k} \longrightarrow M_{-1/2,k} \otimes V$$

We can consider two solutions.

$$T_\pm(q) = \mathrm{Tr}\big|_{M_{\pm 1/2,k}} \left(\phi^\pm(z)\, B\, q^{-\partial} \right)$$

Solving the elliptic KZ, we get

$$T_\pm(q) = \eta(q^2)^{3/4k}\, v_\pm$$

$$\eta(q) = q^{1/24} \prod_{n\geq 1} (1-q^n) \qquad \text{(Dedekind } \eta\text{-fct)}$$

16. Fundamental solution

Let V be a finite dimensional $\mathfrak{g}$-module, $u \in V$ a vector of weight μ. Let $\phi^\mu(z) : M_{\lambda,k} \longrightarrow M_{\lambda+\mu,k} \otimes V(z)$ be the intertwiner for which

$$\langle v^*_{\lambda+\mu,k},\ \phi^\mu(z)\, v_{\lambda,k} \rangle = z^\Delta u\,(1 + O(z))$$
$$z \to 0$$

<u>Def 1</u> The fundamental solution of the elliptic KZ equations is the $\mathrm{End}(V_1 \otimes \cdots \otimes V_n)$-valued function $\mathcal{F}$ of $z_1, \dots, z_n, q$ such that if $u = u_1 \otimes \cdots \otimes u_n \in V_1 \otimes \cdots \otimes V_n$ is a vector of weight μ, then

$$\mathcal{F}(z_1, \dots, z_n | q)\, u = \mathrm{Tr}\big|_{M_{\lambda_0, k}} \left(\phi^{u_1}(z_1) \cdots \phi^{u_n}(z_n) B q^{-\partial}\right)$$

where $\lambda_0 = (\beta - 1)^{-1}(\mu)$.

<u>Def 2</u> Fundamental solution is the solution $\mathcal{F}$ with values in $\mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ which has the property $\mathcal{F} \sim z_1^{D_1} \cdots z_n^{D_n}$, $q \to 0$, $z_j / z_{j+1} \to \infty$, where $D_j \in \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ are defined by

$$D_j(u_1 \otimes \cdots \otimes u_n) = \frac{\langle \lambda_{j-1}, \lambda_{j-1} \rangle - \langle \lambda_j, \lambda_j \rangle}{2(k+1)}\, u_1 \otimes \cdots \otimes u_n$$

$$\lambda_j = (\beta - 1)^{-1}\Big(\sum_{i=1}^{n} x_i\Big) + \sum_{i=1}^{j} x_i$$

x_j are the weights of $u_j \in V_j$.

These definitions are equivalent.

17. Modular invariance

Fact: The y_j-part of the elliptic KZ system is modular invariant, i.e., does not change under

$$\hat{y}_i = \frac{y_i}{c\tau+d}, \qquad \hat{\tau} = \frac{a\tau+b}{c\tau+d}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$$

$$\Gamma(N) = \{ A \in SL_2(\mathbb{Z}) \mid A \equiv Id \mod N \}$$

(congruence subgroup)

However, the τ-equation is not completely invariant. If you apply $A \in \Gamma(N)$, you have to subtract a linear combination of y_j-equations with suitable coefficients, and then you get almost the original equation:

$$\varkappa \frac{\partial F}{\partial \tau} = \sum_{i,j=1}^{n} L_{ij}(\tau) F + \sum_{j<i} S_{ij}(y_i - y_j | \tau) F + \sum_{i=1}^{n} \frac{c\,\delta_i}{2N(c\tau+d)} F,$$

where δ_i is the value of the Casimir of $\mathfrak{g}$ in V_i. Therefore, we have a corresponding invariance statement about solutions.

claim The fundamental solution $\mathcal{F}$ transforms under a modular transformation $A \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$ as follows:

$$\mathcal{F}(\hat{y}_1, \cdots, \hat{y}_n \mid \hat{\tau}) = (c\tau+d)^{\frac{1}{2N\varkappa}\sum_{i=1}^{n}\delta_i} \times$$
$$\mathcal{F}(y_1, \cdots, y_n \mid \tau)\, X(A)$$

where $X(A)$ is a projective representation of $\Gamma(N)$ in $V_1 \otimes \cdots \otimes V_n$. Thus, our construction assigns to every set of representations $V_1, \ldots, V_n$ of sl_N an action of $\Gamma(N)$ in their tensor product. This action can be suitably extended to an action of $SL_2(\mathbb{Z})$ in $\mathrm{End}(V_1 \otimes \cdots \otimes V_n)$. It is difficult and interesting to compute it.

18. Monodromy

The monodromy of the elliptic KZ system with respect to the z-variable is easily computable since it reduces to interchanging of intertwiners, the same as for the usual KZ equations

For sl_2, we have the "exchange relations"

$$\Phi^{\lambda_1 \lambda_0}(z_1)\, \Phi^{\lambda_2 \lambda_1}(z_2)$$
$$= A^{\pm} \sum_{\nu} R_{\lambda_1 \nu}(\lambda_2, \lambda_0)^{\nu_1 \nu_2}\, \sigma\, \Phi^{\nu \lambda_0}(z_2)\, \Phi^{\lambda_2 \nu}(z_1)$$

where R is the constant quantum R-matrix (without a spectral parameter). Using this relation, we can easily give an explicit formula for what happens to the fundamental solution F when y_i interchanges with y_{i+1} (multiplication by an R-matrix) or when y_i goes around a cycle on the torus $\mathbb{C}/\langle 1, \tau\rangle$. (a product of R-matrices and diagonal matrices). Thus, we can compute the monodromy group of the elliptic KZ equations. Note that this group will not be (in general) the braid group of the torus but its central extension, since the elliptic KZ equations are not a local system on $(\mathbb{C}/\langle 1, \tau\rangle)^n$ but only a projective local system. It has to do with the fact that although the automorphisms β and γ of sl_N defined earlier commute with each other, the corresponding matrices B and C in GL_N do not. They generate a Heisenberg group of order N^3. This group plays an important role in the theory of the elliptic KZ equations, in particular, it gives rise to a 2-cocycle on the braid group.

It is interesting that if all V_i are fundamental representations, and n, N are coprime, the monodromy representation of the extended braid group of the torus arising from the elliptic KZ equation factors through the Cherednik "Double Affine Hecke Algebra".